SOME FIBERED AND NON-FIBERED LINKS AT INFINITY OF HYPERBOLIC COMPLEX LINE ARRANGEMENTS

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ABSTRACT. Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$, $d := \dim_{\mathbb{R}}(\mathbb{F})$. Denote by $\mathcal{P}(\mathbb{F})$ either the affine plane $\mathcal{A}(\mathbb{F})$ or the hyperbolic plane $\mathcal{H}(\mathbb{F})$ over $\mathbb{F}$. An arrangement $\mathcal{L}$ of k lines in $\mathcal{P}(\mathbb{F})$ (pairwise non-parallel in the hyperbolic case) has a link at infinity $K_{\infty}(\mathcal{L})$ comprising k unknotted $(d-1)$-spheres in S^{2d-1}, whose topology reflects the combinatorics of $\mathcal{L}$ "at infinity". The class of links at infinity of affine $\mathbb{F}$-line arrangements is properly included in the class of links at infinity of hyperbolic $\mathbb{F}$-line arrangements. Many links at infinity of (essentially non-affine) connected hyperbolic $\mathbb{C}$-line arrangements are far from being fibered. In contrast, if the (affine or hyperbolic) $\mathbb{R}$-line arrangement $\mathcal{L}_{\mathbb{R}} \subset \mathcal{P}(\mathbb{R})$ is connected, and $\mathcal{L} = \mathbb{C}\,\mathcal{L}_{\mathbb{R}} \subset \mathcal{P}(\mathbb{C})$ is its complexification, then $K_{\infty}(\mathcal{L})$ is fibered.

1. INTRODUCTION; STATEMENT OF RESULTS

A link $Q \subset S^3$ is *split* if $\pi_2(S^3 \setminus Q) \neq \{0\}$, and *fibered* if Q has an *open book*, i.e., a map $f : S^3 \to \mathbb{C}$ with $Q = f^{-1}(0)$ such that 0 is a regular value and $f/|f| : S^3 \setminus Q \to S^1$ is a fibration. A fibered link is not split.

Theorem 3.1. *For every knot $K \subset S^3$, for every $t \in \mathbb{Z}$ not greater than the maximal Thurston-Bennequin invariant $\mathrm{TB}(K)$, for all sufficiently large $k \in \mathbb{N}$, there is a connected hyperbolic $\mathbb{C}$-line arrangement whose link at infinity is a t-twisted $+$-clasped chain of k unknots of type K. This link is not split, and it is fibered iff $K = O$ and $t = -1$.*

Theorem 4.1. *The link at infinity of the complexification $\mathbb{C}\mathcal{L}_{\mathbb{R}} \subset \mathcal{P}(\mathbb{C})$ of an $\mathbb{R}$-line arrangement $\mathcal{L}_{\mathbb{R}} \subset \mathcal{P}(\mathbb{R})$ is fibered iff it is not split iff $\mathcal{L}_{\mathbb{R}}$ is connected.*

Section 2 is an introduction to hyperbolic line arrangements and their links at infinity. Theorem 3.1 is proved in Section 3 by "Legendrian

inscription". Theorem 4.1 is proved in Section 4 as an application of A'Campo's method of *divides* [1, 2]. Section 5 contains various remarks.

2. Hyperbolic line arrangements and their links at infinity

2.1. **Affine and hyperbolic lines and planes.** Let $\mathbb{F}$ be $\mathbb{R}$ or $\mathbb{C}$; let $d := \dim_{\mathbb{R}}(\mathbb{F})$. Denote the euclidean norm on the $\mathbb{R}$-vectorspace $\mathbb{F}^2$ by $\| \cdot \|$. Coordinatize the affine plane over $\mathbb{F}$ in the usual way: that is, let $\mathcal{A}(\mathbb{F}) := \mathbb{F}^2$, with its standard structure of $\mathbb{F}$-analytic manifold, and let an *affine $\mathbb{F}$-line* be any translate $(\sigma, \tau) + V$ by $(\sigma, \tau) \in \mathbb{F}^2$ of a 1-dimensional $\mathbb{F}$-subvectorspace $V \subset \mathbb{F}^2$. The *complexification* of the affine $\mathbb{R}$-line $L = (\sigma, \tau) + V$ is the affine $\mathbb{C}$-line $\mathbb{C}L := (\sigma, \tau) + \mathbb{C}V$.

Coordinatize the hyperbolic plane over $\mathbb{F}$ by its Klein model (convenient here, where the hyperbolic metric plays no significant role): that is, let $\mathcal{H}(\mathbb{F}) := \{(\sigma, \tau) \in \mathbb{F}^2 : \|(\sigma, \tau)\| < 1\} = \operatorname{Int} D^{2d}$, with the $\mathbb{F}$-analytic manifold structure induced from $\mathbb{F}^2$, and declare that a *hyperbolic $\mathbb{F}$-line* is any non-empty intersection of $\mathcal{H}(\mathbb{F})$ with an affine $\mathbb{F}$-line. The unique affine $\mathbb{F}$-line containing the hyperbolic $\mathbb{F}$-line $L \subset \mathcal{H}(\mathbb{F})$ will be called the *extension* of L, denoted L^e. The *complexification* of the hyperbolic $\mathbb{R}$-line L is $\mathbb{C}L := \mathcal{H}(\mathbb{C}) \cap \mathbb{C}(L^e)$; of course $(\mathbb{C}L)^e = \mathbb{C}(L^e)$.

As usual in the Klein model, distinct hyperbolic $\mathbb{F}$-lines L_1 and L_2 are *concurrent* if $L_1^e \cap L_2^e \cap \operatorname{Int} D^{2d} \neq \varnothing$, *parallel* if $L_1^e \cap L_2^e \cap \partial D^{2d} \neq \varnothing$, and *hyperparallel* if $L_1^e \cap L_2^e \cap D^{2d} = \varnothing$.

2.2. **Affine and hyperbolic line arrangements.** For present purposes, an *affine $\mathbb{F}$-line arrangement* is the union $\mathcal{L} = \bigcup L_i$ of a (finite) family of pairwise distinct affine $\mathbb{F}$-lines L_i. A *hyperbolic $\mathbb{F}$-line arrangement* is the union $\mathcal{L} = \bigcup L_i$ of a (finite) family of pairwise distinct, pairwise non-parallel hyperbolic $\mathbb{F}$-lines L_i. In either case, a *node* of $\mathcal{L}$ of *degree $m \geq 2$* is a point contained in m lines $L_i \subset \mathcal{L}$; $\mathcal{L}$ is *in general position* if it has no nodes of degree $m > 2$.

Let $\mathcal{L}$ be a hyperbolic $\mathbb{F}$-line arrangement. The *extension* of $\mathcal{L}$ is the affine $\mathbb{F}$-line arrangement $\mathcal{L}^e := \bigcup L_i^e \subset \mathcal{A}(\mathbb{F})$. Easily, every affine $\mathbb{F}$-line arrangement in $\mathcal{A}(\mathbb{F})$ is homothetic to $\mathcal{L}^e$ for some hyperbolic $\mathbb{F}$-line arrangement $\mathcal{L}$. In case $\mathbb{F} = \mathbb{R}$, the *complexification* of $\mathcal{L}$ is the hyperbolic $\mathbb{C}$-line arrangement $\mathbb{C}\mathcal{L} := \bigcup \mathbb{C}L_i$. The *boundary* (resp., *closure*) of $\mathcal{L}$ is $\partial \mathcal{L} := \mathcal{L}^e \cap S^{2d-1}$ (resp., $\overline{\mathcal{L}} := \mathcal{L}^e \cap D^{2d}$); thus $\partial \mathcal{L}$ is the union of a family of k pairwise disjoint $(d-1)$-spheres in S^{2d-1}.

In case $\mathbb{F} = \mathbb{C}$, $\partial \mathcal{L}$ is endowed with a canonical orientation: the orientation of the component $L_i^e \cap S^3 \subset \partial \mathcal{L}$ is that induced on it by the canonical (complex) orientation of L_i^e. In case $\mathbb{F} = \mathbb{R}$, $\partial \mathcal{L}$ is endowed with a canonical partition into 2-sets, namely, $L_i^e \cap S^1$.

Any submanifold of S^{2d-1} isotopic to $\partial\mathcal{L}$, by an ambient isotopy respecting the canonical orientation (resp., partition) in case $\mathbb{F} = \mathbb{C}$ (resp., $\mathbb{F} = \mathbb{R}$), will be called the *link at infinity* of $\mathcal{L}$ and denoted $K_\infty(\mathcal{L})$.

Call a hyperbolic $\mathbb{F}$-line arrangement $\mathcal{L}$ *essentially affine* if $\mathcal{L}^e \setminus \mathcal{L} \subset \mathbb{F}^2 \setminus \mathcal{H}(\mathbb{F})$ is a collar of $\partial\mathcal{L}$ (equivalently, if every node of $\mathcal{L}^e$ is already a node of $\mathcal{L}$). Note that a hyperbolic $\mathbb{R}$-line arrangement is essentially affine iff its complexification is essentially affine. Evidently, affine $\mathbb{F}$-line arrangements are combinatorially and topologically precisely the same as essentially affine hyperbolic $\mathbb{F}$-line arrangements; in particular, the link at infinity of an affine $\mathbb{F}$-line arrangement $\mathcal{L}$ is well-defined (even though $\partial\mathcal{L}$ does not exist), and the notation $K_\infty(\mathcal{L})$ will be used in this case also.

The ambient isotopy type of $K_\infty(\mathcal{L})$ is readily seen to be invariant under a number of operations on $\mathcal{L}$, such as topological equivalence, isotopy, and homotopy (suitably defined), including perturbations which are "sufficiently small". One such invariance in particular will be useful below; its truth depends on the exclusion of parallel lines from hyperbolic $\mathbb{F}$-line arrangements.

Lemma 2.1. *Let $\mathcal{L}$ be a hyperbolic $\mathbb{F}$-line arrangement. For almost all arrangements $\mathcal{L}'$ which are sufficiently close to $\mathcal{L}$ in the euclidean metric on $\mathcal{H}(\mathbb{F})$, $\mathcal{L}'$ is in general position and $K_\infty(\mathcal{L}')$ is ambient isotopic to $K_\infty(\mathcal{L})$.* $\qquad\square$

3. A construction of non-fibered hyperbolic $\mathbb{C}$-line arrangements

3.1. **Legendrian knots and the Thurston-Bennequin invariant of a knot.** At each point of $S^3 \subset \mathbb{C}^2$, the tangent (real) 3-space to S^3 contains a unique affine $\mathbb{C}$-line, the *contact plane* at that point. An unoriented knot $K \subset S^3$ whose tangent line at each point is contained in the contact plane at that point is called *Legendrian*. A Legendrian knot K is framed by a canonical normal linefield (the orthogonal complement of its tangent linefield in the contact planefield); the integer associated in the usual way to that framing is the *Thurston-Bennequin invariant* $\mathrm{tb}(K)$. For any knot $K \subset S^3$, let $\mathrm{TB}(K) := \sup\{\mathrm{tb}(K') : K'$ is Legendrian and ambient isotopic to $K\}$ denote the *maximal Thurston-Bennequin invariant* of K. It is known that $\mathrm{TB}(K)$ is an integer (i.e., $\mathrm{TB}(K) \neq \pm\infty$), and that if $\mathrm{TB}(K) \geq t \in \mathbb{Z}$, then there is a Legendrian knot K' ambient isotopic to K with $\mathrm{tb}(K') = t$. (See [3] and references cited therein.)
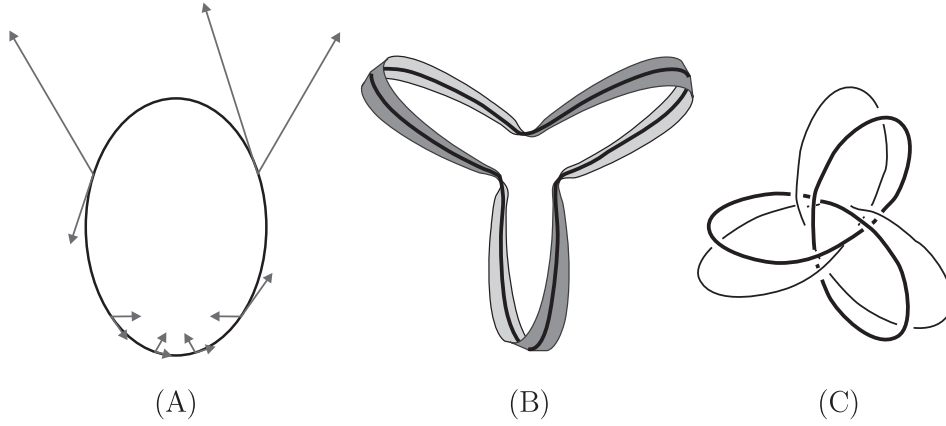
(A) (B) (C)

FIGURE 1. (A) A Legendrian unknot $O\{1, -1\}$ decorated with some tangent vectors and their corresponding Legendrian normal vectors. (B) A Legendrian unknot $O\{1, -3\}$ as the core circle of an annulus $A(O, -3)$. (C) A Legendrian negative trefoil $O\{2, -3\}$ and a parallel obtained by pushing off along the canonical normal linefield; their linking number equals $-6 = 2(-3)$, which is $\mathrm{TB}(O\{2, -3\})$ ([19, Theorem 8]).

Fig. 1 illustrates several Legendrian knots, projected stereographically via

$$S^3 \setminus \{(0, -\mathrm{i})\} \to \mathbb{C} \times \mathbb{R} : (z, w) \mapsto (z, \mathrm{Re}(w))/(1 + \mathrm{Im}(w))$$

and thence orthogonally via pr_1 to $\mathbb{C}$, with their canonical framings depicted in various ways. To simplify the calculations attendant on the preparation of Fig. 1 (and Fig. 3), each knot illustrated has been taken to be of the particularly simple form $S^3 \cap \{2^{-1/2}\tau^p, 2^{-1/2}\tau^{-q}) : \tau \in S^1\}$, where $p, q \in \mathbb{N}$.

3.2. **Annular plumbing and chains of unknots.** Given a knot $K \subset S^3$ and an integer t, denote by $A(K, t) \subset S^3$ any annulus such that $K \subset \mathrm{Int}\, A(K, t)$, K bounds no disk on $A(K, t)$, and the Seifert matrix of $A(K, t)$ is $[t]$ (that is, when $A(K, t)$ is endowed with an orientation, the linking number in S^3 of its two boundary components is $-t$); for example, letting O denote an unknot, $A(O, -1)$ is a *positive Hopf annulus*.

A *transverse arc* of $A(K, t)$ is any arc $\tau \subset A(K, t)$ with $\partial\tau = \tau \cap \partial A(K, t)$ which intersects K transversely and at a single point. Given pairwise disjoint transverse arcs $\tau_i \subset A(K, t)$, $i = 1, \ldots, k$, there are pairwise disjoint 3-disks $N_i \subset S^3$ such that $N_i \cap A(K, t) = \partial N_i \cap$
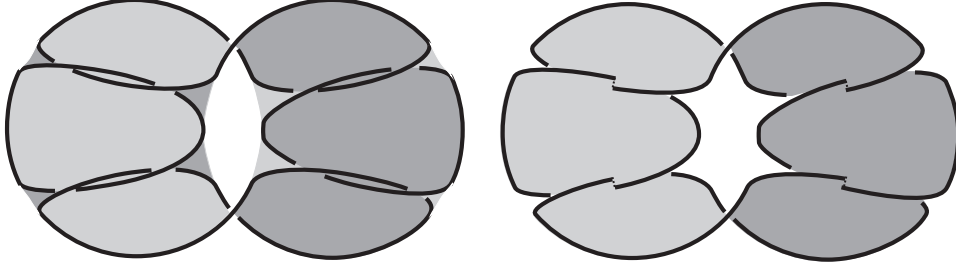
FIGURE 2. Two views of a (-1)-twisted $+$-clasped chain of 4 unknots of type O.

$A(K, t)$ is a regular neighborhood of τ_i on $A(K, t)$, and positive Hopf annuli $A(O_i, -1) \subset N_i$ such that $A(K, t) \cap \partial N_i = A(O_i, -1) \cap \partial N_i = A(O_i, -1) \cap A(K, t)$ and $K \cap A(O_i, -1)$ is a transverse arc of $A(O_i, -1)$. The union

$$F(K, t, k) := A(K, t) \cup \bigcup_{i=1}^{k} A(O_i, -1) \subset S^3$$

is an orientable surface; it is, in fact, an iterated *plumbing* (quadrilateral Murasugi sum) of annuli [6, 22].

As is easily seen, $F(K, t, k)$ equipped with either orientation is ambient isotopic to $F(K, t, k)$ equipped with the opposite orientation, so its boundary $\partial F(K, t, k)$ has an orientation which is well-defined up to ambient isotopy. For $k \geq 2$, $\partial F(K, t, k)$ is a link of k components; each component is an unknot, and any 2-component sublink is either a trivial link (if it is split) or isotopic to $\partial A(O, -1)$ (if it is not split). Call $\partial F(K, t, k)$ a *t-twisted $+$-clasped chain of k unknots of type K*.

Fig. 2 illustrates two views of a (-1)-twisted $+$-clasped chain of 4 unknots of type O: at the left, it is shown bounding the iterated plumbing of annuli $F(O, -1, 4)$; at the right, it is shown bounding the union of four individually embedded 2-disks immersed in S^3 with (positive) clasp singularities.

Theorem 3.1. *For every knot $K \subset S^3$, for every $t \in \mathbb{Z}$ not greater than $\mathrm{TB}(K)$, for all sufficiently large $k \in \mathbb{N}$, there is a connected hyperbolic $\mathbb{C}$-line arrangement $\mathcal{L}$ such that $K_\infty(\mathcal{L}) = \partial F(K, t, k)$. This link is fibered iff $K = O$ and $t = \mathrm{TB}(O) = -1$.*

Proof. Without loss of generality, K is Legendrian and $t = \mathrm{tb}(K)$. Given $p, q \in S^3$ with $p \neq q$, let $M(p, q) \subset \mathbb{C}^2$ denote the affine $\mathbb{C}$-line through p and q; let $M(p, p)$ denote the contact plane at p. Thus $M(p, q) \cap S^3$ is a (round) circle for $p \neq q$, and $M(p, p) \cap S^3$ reduces

to the singleton $\{p\}$. Let $d(p, q)$ denote the (euclidean) diameter of $M(p, q) \cap S^3$. Because K is Legendrian, the limit of $d(p, q)$ as $q \in K$ approaches $p \in K$ along K exists and is $d(p, p) = 0$. It follows that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $p, q \in K$ and $\|p - q\| < \delta$, then $d(p, q) < \varepsilon$. Consequently, for all sufficiently large k, there is a sequence of k points $p_1, \ldots, p_k, p_{k+1} = p_1$ on K (cyclically ordered identically by their indices and by their position on K) and a piecewise-smooth annulus $A(K, t)$ with $A(K, t) \supset M(p_i, p_{i+1}) \cap S^3$ for all i. Denote by $\widetilde{L}_i$ the hyperbolic $\mathbb{C}$-line $M(p_i, p_{i+1}) \cap \mathcal{H}(\mathbb{C})$. The union $\widetilde{\mathcal{L}} := \bigcup_{i=1}^k \widetilde{L}_i$ (which is, as it were, an "inscribed hyperbolic $\mathbb{C}$-polygon" of K) is not a hyperbolic $\mathbb{C}$-line arrangement as defined in this paper, since $\widetilde{L}_i$ and $\widetilde{L}_{i+1}$ are parallel; but almost any small perturbation of $\widetilde{\mathcal{L}}$ will be a hyperbolic $\mathbb{C}$-line arrangement, and for a suitable such perturbation $\mathcal{L}$ (easily achieved by moving all the affine $\mathbb{C}$-lines $M(p_i, p_{i+1})$ slightly closer to the origin) the link at infinity $K_\infty(\mathcal{L})$ is isotopic to $\partial F(K, t, k)$.

It remains to be shown that, with K and t as above, $\partial F(K, t, k)$ is fibered iff $K = O$ and $t = -1$. This can be done by invoking results about quasipositivity, Murasugi sums, and fiber surfaces. The annulus $A(K, t)$ is quasipositive (by [19]), as is each $A(O_i, -1)$, so (by [22]) the plumbed surface $F(K, t, k)$ is quasipositive and therefore a least-genus surface for its boundary. As is well-known, a least-genus Seifert surface bounded by a fibered link must be a fiber surface. By Gabai [6], a Murasugi sum of Seifert surfaces is a fiber surface iff all of the summands are fiber surfaces. Since $A(O_i, -1)$ is indeed a fiber surface, $\partial F(K, t, k)$ is fibered iff $A(K, t)$ is a fiber surface. The only annuli which are fiber surfaces are $A(O, -1)$ and its mirror image $A(O, 1)$. Since $\mathrm{TB}(O) = -1$ (cf. [21]), $A(K, t)$ must be $A(O, -1)$. $\qquad\square$

Fig. 3 illustrates the application of Theorem 3.1 to the knots in Fig. 1.

4. DIVIDES AND FIBERED HYPERBOLIC $\mathbb{C}$-LINE ARRANGEMENTS

Following A'Campo [1, 2], let a *divide* be the image $\iota(J)$ of a non-empty compact 1-manifold-with-boundary J by a smooth immersion $\iota : J \to D^2$ such that

1. ι is proper, that is, $\iota^{-1}(\partial D^2) = \partial J$;
2. ι is transverse to ∂D^2;
3. $\iota(J)$ is connected;
4. $\iota(J)$ has only finitely many singular points, each of which is a doublepoint with distinct tangent lines.

A'Campo gives a simple, beautiful construction (essentially a "unoriented Gaussian resolution" of ι), phrased in terms of the (co)tangent

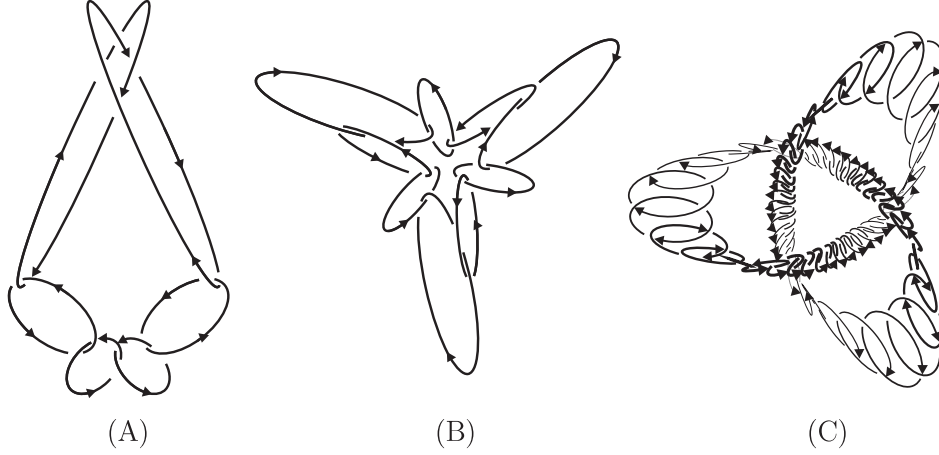(A)   (B)   (C)

FIGURE   3. (A) $\partial F(O, -1, 6)$.   (B) $\partial F(O, -3, 9)$.
(C) $\partial F(O\{2, -3\}, -6, 72)$.

bundle of D^2, which associates to any divide $\iota(J) \subset D^2$ a link $K(\iota(J)) \subset S^3$. The components of $K(\iota(J))$ are in natural one-to-one correspondence with the components of J, and there is a natural orientation for $K(\iota(J))$ (up to simultaneous reversal of orientation on all components). A'Campo then proves that $K(\iota(J))$ is a fibered link.

   Fig. 4 illustrates A'Campo's construction. (The image of the circle $\partial\mathcal{H}(\mathbb{R}) \subset S^3$ by the stereographic projection in 3.1 lies in $\mathbb{R} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$, and thus projects via $\mathrm{pr}_1 : \mathbb{C} \times \mathbb{R} \to \mathbb{C}$ onto a line segment; in Fig. 4, the diagram of the fibered link has been produced using the alternative projection $\mathrm{Re} \times \mathrm{pr}_2 : \mathbb{C} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, allowing it to be legibly supplemented with a diagram of $\partial\mathcal{H}(\mathbb{R})$.)

Theorem 4.1. *The link at infinity of the complexification* $\mathbb{C}\mathcal{L}_\mathbb{R} \subset \mathcal{P}(\mathbb{C})$ *of an* $\mathbb{R}$-*line arrangement* $\mathcal{L}_\mathbb{R} \subset \mathcal{P}(\mathbb{R})$ *is fibered iff* $\mathcal{L}_\mathbb{R}$ *is connected.*

Proof. If $\mathcal{L}_\mathbb{R} \subset \mathcal{H}(\mathbb{R})$ is not connected, then $K_\infty(\mathbb{C}\mathcal{L}_\mathbb{R})$ is a split link and so not fibered. If $\mathcal{L}_\mathbb{R}$ is connected, then (by Lemma 2.1) there is a small perturbation $\mathcal{L}'_\mathbb{R}$ of $\mathcal{L}_\mathbb{R}$ such that $K_\infty(\mathcal{L}')$ and $K_\infty(\mathcal{L})$ are ambient isotopic and $\mathcal{L}'_\mathbb{R}$ is in general position; of course $\mathcal{L}'_\mathbb{R}$ is also connected, so its closure $\overline{\mathcal{L}'_\mathbb{R}}$ is a divide. An inspection of A'Campo's construction in [1], translating his symplectic approach into complex language, immediately reveals that $K(\overline{\mathcal{L}'_\mathbb{R}})$ (with one of its two natural orientations) and $K_\infty(\mathbb{C}\mathcal{L}'_\mathbb{R})$ are identical, whence $K_\infty(\mathbb{C}\mathcal{L}_\mathbb{R})$ is fibered.
   The affine case reduces immediately to the hyperbolic case. $\square$
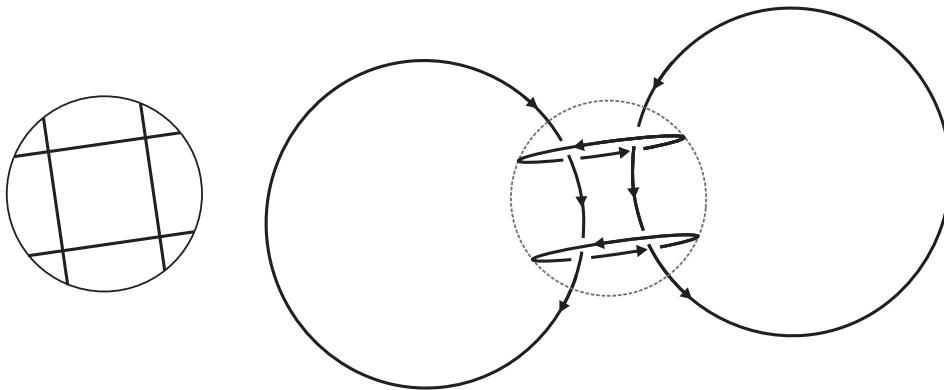
FIGURE 4. A divide which is the closure of an essentially affine hyperbolic $\mathbb{R}$-line arrangement; A'Campo's fibered link of this divide (which is also the link at infinity of the arrangement, and isotopic to the link in Fig. 2), supplemented with the circle $\partial\mathcal{H}(\mathbb{R}) \subset \partial\mathcal{H}(\mathbb{C}) = S^3$.

5. REMARKS

5.1. **Remarks on Section 2.** (1) As combinatorial structures, links at infinity of hyperbolic $\mathbb{R}$-line arrangements are more or less identical to the "chord diagrams" used in the theory of Vassiliev invariants. The significance of this coincidence, if any, is unclear.

(2) The link at infinity $K_\infty(\Gamma) \subset S^3$ of an affine $\mathbb{C}$-algebraic curve $\Gamma \subset \mathbb{C}^2$, introduced in [17], has been studied quite extensively (see, e.g., [12, 14, 13, 8, 10, 16, 5, 4]; see also [11]). There are strong topological restrictions on $K_\infty(\Gamma)$, and in particular $K_\infty(\Gamma)$ is solvable and "nearly fibered" (see, e.g., [13] and [11]). Typically, in $\mathbb{C}^2$, if Γ is an affine $\mathbb{C}$-analytic curve which is not $\mathbb{C}$-algebraic, then the isotopy type of $(1/r)(\Gamma \cap S_r^3) \subset S^3$ does not stabilize as $r \to \infty$, and so there is no apparent useful notion of the link at infinity of such a curve. By contrast, in $\mathcal{H}(\mathbb{C})$, where there is no obvious special class of $\mathbb{C}$-analytic curves which might usefully be called "algebraic", there is nonetheless a plentiful supply of $\mathbb{C}$-analytic curves (among them, curves equivalent by suitable diffeomorphisms $\mathcal{H}(\mathbb{C}) \to \mathbb{C}^2$ to arbitrary $\mathbb{C}$-algebraic curves in $\mathbb{C}^2$) with well-defined links at infinity. Such curves were studied from a topological viewpoint in, e.g., [18, 20, 21], as were the corresponding links (under such names as *transverse* $\mathbb{C}$-*links*), which turn out to be topologically much more diverse than in the affine case (for instance, they are rarely solvable, and typically they are very far from being fibered); however, the interpretation in terms of $\mathcal{H}(\mathbb{C})$ has heretofore
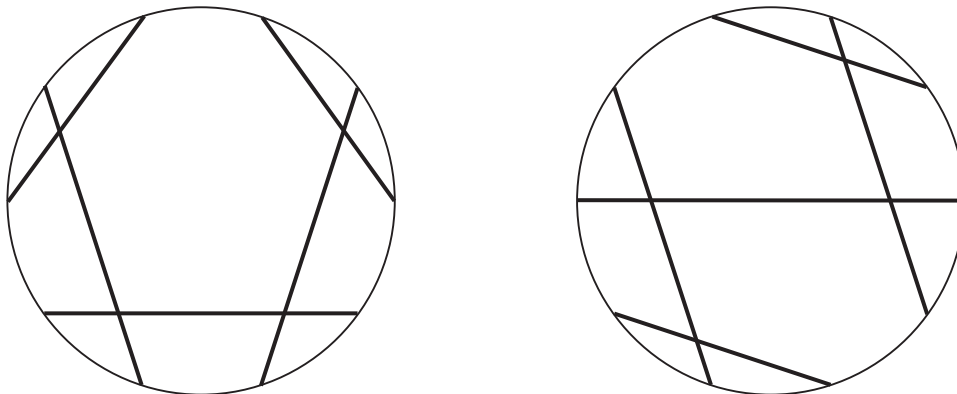
FIGURE 5. There is no line in the hyperbolic $\mathbb{R}$-line arrangement $\mathcal{L}_{\mathbb{R}}$ (left) whose boundary separates the boundaries of two other lines in $\mathcal{L}_{\mathbb{R}}$. The boundary of the horizontal line in $\mathcal{L}'_{\mathbb{R}}$ (right) does separate the boundaries of two other lines in $\mathcal{L}'_{\mathbb{R}}$. Thus $K_{\infty}(\mathcal{L}_{\mathbb{R}})$ and $K_{\infty}(\mathcal{L}'_{\mathbb{R}})$ are not ambient isotopic, though $K_{\infty}(\mathbb{C}\mathcal{L}_{\mathbb{R}}) = K_{\infty}(\mathbb{C}\mathcal{L}'_{\mathbb{R}})$.

been completely overlooked. I hope to come back soon to a new study of general links at infinity in $\mathcal{H}(\mathbb{C})$, in which the techniques of [18] (etc.) will be supplemented with techniques of hyperbolic geometry.

(3) Hyperbolic $\mathbb{R}$-line arrangements with non-isotopic links at infinity can easily have complexifications with isotopic links at infinity. For instance, if $\mathcal{L}_{\mathbb{R}} = \bigcup_{i=1}^{k} L_i$ is a "line tree"—that is, $\mathcal{L}_{\mathbb{R}}$ is in general position and contractible (equivalently, the abstract 1-complex $\Gamma(\mathcal{L}_{\mathbb{R}})$ dual to $\mathcal{L}_{\mathbb{R}}$, whose vertices are $L_1, \ldots, L_k$ and whose edges are the nodes of $\mathcal{L}_{\mathbb{R}}$, is a tree)—then the ambient isotopy type of $K_{\infty}(\mathbb{C}\mathcal{L}_{\mathbb{R}})$ depends only on $\Gamma(\mathcal{L}_{\mathbb{R}})$ (it is a "$\Gamma(\mathcal{L}_{\mathbb{R}})$-connected sum" of $k-1$ copies of $\partial A(O, -1)$); however, for $k > 4$ there exist pairs $\mathcal{L}_{\mathbb{R}}$, $\mathcal{L}'_{\mathbb{R}}$ with $K_{\infty}(\mathcal{L}_{\mathbb{R}}) \neq K_{\infty}(\mathcal{L}'_{\mathbb{R}})$ and $\Gamma(\mathcal{L}_{\mathbb{R}}) = \Gamma(\mathcal{L}'_{\mathbb{R}})$. Fig. 5 gives an example.

5.2. **Remarks on Section 3.** (1) The word "chain" is already used in complex hyperbolic geometry (see, e.g., [7]) to refer to certain important real curves in S^3 (and more generally S^{2n-1}, considered as the boundary of complex hyperbolic n-space), which are not chains as defined here. On the other hand, chains as defined here (specifically, chains of type O) have been studied in the context of 3-dimensional real hyperbolic geometry by Neumann and Reid [15]. Neither of these coincidences should cause confusion.

(2) It seems likely that the following generalization of Theorem 3.1 can be established by similar techniques.

Conjecture 5.1. *For every quasipositive Seifert surface $F \subset S^3$, there exists a system of pairwise disjoint proper arcs $\alpha_1, \ldots, \alpha_\mu \subset F$ such that, for all sufficiently large $k_1, \ldots, k_\mu \in \mathbb{N}$, if F' is the result of plumbing k_i copies of $A(O, -1)$ to F along parallels of α_i for $i = 1, \ldots, \mu$, then $\partial F'$ is isotopic to the link at infinity of a hyperbolic $\mathbb{C}$-line arrangement.*

The second half of the proof of Theorem 3.1 would still apply, showing that for such a surface F', $\partial F'$ is fibered iff ∂F is fibered.

5.3. **Remark on Section 4.** A modified (and strengthened) version of Theorem 4.1 may be proved using other techniques (and other language), notably those of [23]. Let $\mathcal{L}_{\mathbb{R}} \subset \mathcal{H}(\mathbb{R})$ be a connected hyperbolic $\mathbb{R}$-line arrangement.

Theorem 5.2. *If $\mathcal{L}_{\mathbb{R}} \subset \mathcal{H}(\mathbb{R})$ is essentially affine, then $K_\infty(\mathbb{C}\,\mathcal{L}_{\mathbb{R}})$ is a closed positive braid. In any case, there exists an espalier $\mathcal{T}$ such that $K_\infty(\mathbb{C}\,\mathcal{L}_{\mathbb{R}})$ is a closed $\mathcal{T}$-positive braid.*

The proof will be given elsewhere.

5.4. **Further remarks.** As mentioned in Section 1, an *open book* for an oriented link $K \subset S^3$ is a smooth map $f : S^3 \to \mathbb{C}$ with $K = f^{-1}(0)$, such that 0 is a regular value of f and $f/|f| : S^3 \setminus f^{-1}(0) \to S^1$ is a fibration; and an oriented link K is *fibered* if there is an open book for K. A *trivial unfolding*, as defined in [12] (see also [9]), is more or less the cone on an open book. For example, if $H(k)$ is the fibered link comprised of $k \geq 1$ coherently oriented fibers of the standard positive Hopf fibration $S^3 \to S^2$ (so $H(1) = O$ and $H(2) = \partial A(O, -1)$), then the open book $f_k : S^3 \to \mathbb{C} : (z, w) \mapsto z^k + w^k$ for $H(k)$ yields the trivial unfolding $u_k : \{(z, w) \in D^4 : |z^k + w^k| \leq \varepsilon\} \to \mathbb{C} : (z, w) \mapsto z^k + w^k$ (for any sufficiently small $\varepsilon > 0$). The domain D of a trivial unfolding u is a smooth 4-ball-with-corners, the corners of D being the boundary of a tubular neighborhood $N_{\partial D}(u^{-1}(0))$, and the restriction $u|\partial D$ being an open book (modulo a corner-smoothing identification of ∂D with S^3).

Generalizing the definition of trivial unfolding, in [12] (see also [11]) an *unfolding* is defined as a map $u : D \to \mathbb{C}$, where D is a smooth 4-ball-with-corners, $u|\partial D$ is an open book (modulo corner-smoothing), and $u|\operatorname{Int} D$ has only finitely many critical points $\mathbf{x}_i, i = 1, \ldots, m$, near which it restricts to trivial unfoldings (in appropriately adapted coordinates) $u_i : D_i \to \mathbb{C}$. Write $K = \Upsilon_{i=1}^m K_i$ to indicate that some such unfolding, with local fibered links K_i, exists; this symbolic equation suppresses the fact that $K_1, \ldots, K_m$ by no means determine K (for $m > 1$).

Let $\mathcal{L}_\mathbb{R}$ be a connected hyperbolic $\mathbb{R}$-line arrangement with n nodes, of degrees $k_1, \ldots, k_n$. Among the components of $\mathcal{H}(\mathbb{R}) \setminus \mathcal{L}_\mathbb{R}$, let there be exactly s which are hyperbolically bounded (that is, have closure contained in $\mathcal{H}(\mathbb{R})$). Let $k_{n+1} = \cdots = k_{n+s} = 2$.

Theorem 5.3. $K_\infty(\mathbb{C}\,\mathcal{L}_\mathbb{R}) = \Upsilon_{i=1}^{n+s} H(k_i)$.

For essentially affine $\mathcal{L}_\mathbb{R}$, this follows from [12] (or [11]). Various proofs can be given for arbitrary (connected) $\mathcal{L}_\mathbb{R}$: one uses Theorem 5.2 in combination with techniques of [23]; another proceeds (reasonably directly) from A'Campo's results [1, 2]; a third, using the techniques (rather than just the language) of knot-theoretical unfolding, and extending the statement of the theorem considerably, will be given in [24].

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